Scaling of clusters and winding-angle statistics of isoheight lines in two-dimensional Kardar-Parisi-Zhang surfaces

A. A. Saberi[1,*] and S. Rouhani[2]

[1]*School of Physics, Institute for Research in Fundamental Sciences (IPM), 19395-5531 Tehran, Iran*

[2]*Department of Physics, Sharif University of Technology, P.O. Box 11155-9161, Tehran, Iran*

(Received 19 November 2008; published 11 March 2009)

We investigate the statistics of isoheight lines of (2+1)-dimensional Kardar-Parisi-Zhang model at different level sets around the mean height in the saturation regime. We find that the exponent describing the distribution of the height-cluster size behaves differently for level cuts above and below the mean height, while the fractal dimensions of the height-clusters and their perimeters remain unchanged. The statistics of the winding angle confirms the previous observation that these contour lines are in the same universality class as self-avoiding random walks.

DOI: 10.1103/PhysRevE.79.036102 PACS number(s): 68.35.Ct, 89.75.Da, 61.05.−a, 11.25.Hf

The study of surfaces, their static statistical properties, as well as growth and evolution dynamics and their morphological properties, has been attracting an ever increasing amount of interest over the last two decades.

A method of characterizing surfaces is by looking at their isoheight contour lines and the islands that are generated by a cut through the surface at a certain constant height. This is, for example, observed in the patterns exhibited by topographical islands and continents. These islands and their coastlines have some fractal features with a fractal dimension related to the roughness exponent α, given by the structure function $\langle[h(x)-h(x+r)]^2\rangle \sim r^{2\alpha}$, where $h(x)$ is the height of interface as a function of its position.

Theoretical modeling of the growth processes started with the work by Edwards and Wilkinson (EW) [1] who suggested that one might describe the dynamics of the height fluctuations by a simple linear diffusion equation. Kardar, Parisi, and Zhang (KPZ) [2] realized that there is a relevant term proportional to the square of the height gradient which represents a correction for lateral growth. The KPZ equation is given by

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}|\nabla h|^2 + \eta(\mathbf{x},t). \tag{1}$$

The first term on the right—hand side describes relaxation of the interface caused by a surface tension ν, and the nonlinear term is due to the lateral growth. The noise η is uncorrelated Gaussian white noise in both space and time with zero average i.e., $\langle\eta(\mathbf{x},t)\rangle=0$ and $\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle=2D\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')$.

In (1+1) dimension the roughness and growth exponents were exactly obtained, $\alpha=1/2$ and $\beta=1/3$, respectively [3], while for the (2+1)-dimensional case there are just numerical evidence and predictions [4].

The KPZ equation is invariant under translations along both growth direction and perpendicular to it, as well as time translation and rotation. However, the existence of the nonlinear term breaks the up-down symmetry $(h\to-h)$ [3]. In two dimensions these symmetries and growth dynamics can affect the statistics of the isoheight lines (island coastlines) at different level sets, which is the main subject of the present work.

In our previous paper [5] we have focused on zero isoheight lines of the (2+1)-dimensional KPZ model in the saturation regime (mean height was set to zero). Using the theory of Schramm-Loewner evolution (SLE), we have numerically shown that the contour lines of zero height behave statistically like self-avoiding walks (SAWs) and they can be defined by the family of conformally invariant curves i.e., SLE_κ curves with diffusivity $\kappa=8/3$. The statistics of these objects for the EW model has been shown to be the same as the interfaces in the $O(2)$ model, and can be described by SLE_4.

The SLE process, introduced by Schramm [6], describes the scaling limit of a variety of statistical mechanical models in two dimensions (some review articles are cited in [7]). Schramm and Sheffield [8] showed that the contour lines in a two-dimensional discrete Gaussian free field are statistically equivalent to SLE_4. Moreover, it is shown that the restriction property only applies in the case for $\kappa=8/3$ [9]. Since self-avoiding random walk (SAW) satisfies the restriction property, it is conjectured that in the scaling limit it falls in the SLE class with $\kappa=8/3$ [10]. The scaling limit of SAW in the half-plane has been proven to exist [11] but there is no general proof of its existence.

The theory of SLE has recently been applied to many experimental and physical systems. It is shown that the statistics of the zero-vorticity lines in inverse cascade of two-dimensional (2D) Navier-Stokes turbulence is conformally invariant and belongs to the percolation universality class [12]. The same issue has been studied for zero-temperature isolines in the inverse cascade of surface quasigeostrophic turbulence [13], domain walls of spin glasses [14], and the nodal lines of random wave functions [15]. Moreover, it has been shown recently that the statistics of the isoheight lines on the experimentally grown WO_3 surface is the same as domain walls statistics in the critical Ising model [16]. Avalanche frontiers in sandpile models have been shown to be conformally invariant and in the same universality class of loop erased random walks [17].

*a_saberi@ipm.ir

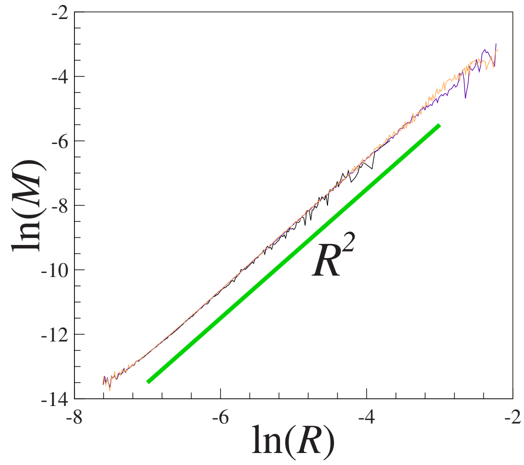

FIG. 1. (Color online) Log-log plot of the average area of a cluster M versus the radius of gyration R, at three different cuts with δ=−2, 0, and +2.

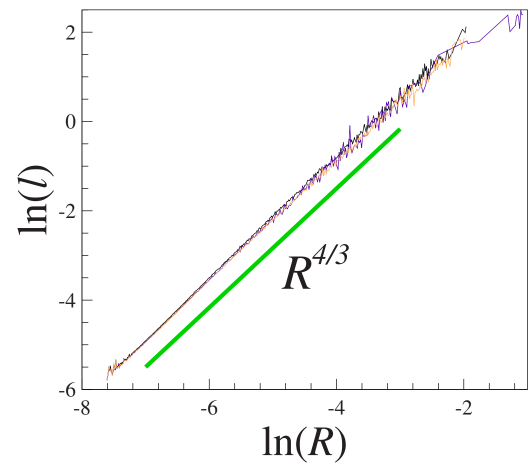

FIG. 2. (Color online) Log-log plot of the average length of a loop l versus the radius of gyration R, at three different cuts with δ=−2, 0, and +2.

Here we briefly review the results obtained in [5] for zero height level cuts, and then extend them for cuts made at different heights.

We have integrated the discretized KPZ equation on a square lattice of size 2048^2 with periodic boundary conditions. The details of numerical integration and simulation are given in [5].

Consider an ensemble of 2D-KPZ saturated surfaces and a cut is made at specific height say $h_\delta = \langle h(x)\rangle + \delta\sqrt{\langle [h(x) - \langle h(x)\rangle]^2\rangle} := 0$, where the symbol $\langle\cdots\rangle$ denotes spatial averaging. Then we define each island (cluster height) as a set of connected sites with positive height which were identified by the Hoshen-Kopelman algorithm.

The scaling of the mass M of a cluster with the radius of gyration R, behaves like $M \sim R^{D_c}$, where D_c is the fractal dimension of the cluster which is D_c=2 in this case. As shown in Fig. 1, this fractal dimension remains unchanged for different δ.

The fractal dimension of a coastline (or loop because of periodic boundary conditions) can be obtained with the scaling relation between the average length of a loop l, and the radius of gyration R, as $l \sim R^{D_l}$. This fractal dimension remains also fixed, within numerical errors, for cuts made at different δ. The best fits to data shown in Fig. 2 yield the fractal dimension of a contour line in the range of $D_l = 1.34 \pm 0.02$ (Fig. 2).

Powerful scaling arguments made by Kondev and Henley [18] connect the fractal dimension of a contour line to the roughness exponent α of the surface,

$$D_l = 2 - x_l - \alpha/2, \qquad (2)$$

where x_l is the loop correlation exponent. Although the exact value of x_l=1/2 is for α=0 and 1, [19] but it is conjectured that its value is super universal and is independent of α for Gaussian surfaces.

In the case of 2D-KPZ surface the finite-size scaling for the interface width yields the roughness exponent [5] $\alpha = 0.37 \pm 0.01$, which is in mild conflict of Eq. (2). This may be because the field $h(x)$ does not follow a Gaussian distribution. In other words, the fractal dimension $D_l = 1.34 \pm 0.02 \sim 4/3$ obtained for contour lines of 2D-KPZ surface is equal to what one obtains for Gaussian surfaces with α=1/3.

The island-size distribution has also a power-law behavior. As can be seen in Fig. 3, there are two distinct scaling regions for the distribution of the island size. We find that the small size islands are distributed according to a power-law distribution $n(M) \sim M^{-\tau_s}$, with a same exponent $\tau_s = 2 \pm 0.05$ for level cuts at different δ. For the other region with larger island size dominant, the power-law behavior is held, but with different exponents at different level cuts.

For cuts made at lower values of δ, percolative height clusters appear dominantly which their sizes are in the order of sample size. Inside these percolative islands there are some lakes (negative height clusters) which can also contain

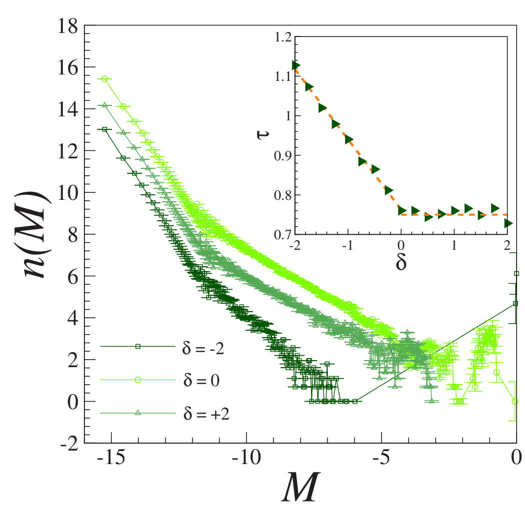

FIG. 3. (Color online) Main: log-log plot of the number of clusters of area between M and $1.05M$, at three different cuts with δ=−2, 0, and +2. Inset: the exponents for the island distributions as a function of δ. The errors are less than 0.05 for all exponents. The slope of the dashed lines is -1.77 ± 0.03, the best fitted for $\delta < 0$, and 0, which is drawn for comparison for $\delta \geq 0$ at τ=0.75.

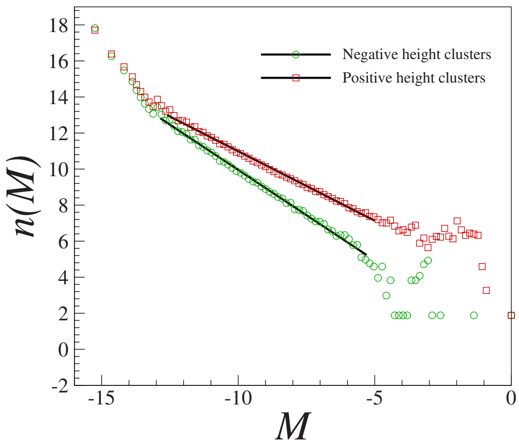

FIG. 4. (Color online) Log-log plot of the number of positive (squares) and negative (circles) height-clusters of area between M and $1.1M$, at a level cut made at the mean height with $\delta=0$. Solid lines are the best fits to the data in the linear region yielding the distribution exponent $\tau\sim 0.77$ and $\tau\sim 1$ for positive and negative height-clusters, respectively.

smaller islands with the distribution as shown in Fig. 3 (for $\delta=-2$). Increasing in the height of the cut tends to a continuous distribution of islands with different sizes and different scaling behavior. We find that for cuts below the mean height (i.e., $\delta<0$), the island-size distribution exponent τ takes different values for different δ, while it takes almost a same value (within statistical error) for positive height cuts. This behavior is shown in the inset of Fig. 3, which attributes two different regimes for level cuts i.e., $\delta<0$ and $\delta\geq 0$. For level cuts made at $\delta<0$, the island-size distribution exponent τ decreases linearly with the slope of -1.77 ± 0.03, while it takes values around $\tau=0.75$ within statistical errors for $\delta\geq 0$. This asymmetrical behavior of the exponent τ around the mean height may be interpreted as the breakdown of the up-down symmetry under changing $h\rightarrow -h$ in the KPZ [Eq. (1)]. This asymmetry in the dynamics of the growth process can tend to an asymmetry in the statistics and the distribution of valleys and overhangs [20].

In a system with up-down symmetry it is expected that the valleys and overhangs have the same statistical behavior with same distribution. This behavior is not seen in the mean-height cuts of the 2D-KPZ surface. The size distribution of the islands with positive and negative heights differs. As shown in Fig. 4, the exponents defining these two distributions are quite different with values $\tau\sim 0.77$ and $\tau\sim 1$ for positive and negative height clusters, respectively.

Although the exact cause for the behavior of the exponent τ as function of the height of the cut eludes us, but it may be possible to guess its behavior around the mean height [18] (a slightly different problem has been investigated in [21] for random Gaussian surfaces).

Using scaling arguments, it is shown in [18] that the average number density of contour lines (the coastlines here), scales with the radius of gyration R as $n(R)\sim R^{-2+\alpha}$. Since the dimension of the islands in our case is 2 (see Fig. 1), one can expect that the radius of gyration for the perimeter of the islands and the islands themselves have the same statistical behavior. So, the average number density of the islands of size M, is given by $n(M)\sim M^{-1+\alpha/2}$. Within an uncertainty in determining the roughness exponent α, it yields an approximated value for the exponent $\tau\sim 0.81$ which works here for the height cuts around and above the mean height.

A study similar to ours was done in [21] for random Gaussian surfaces with a slightly different approach in defining the islands (they consider the area ratio of the cut instead of the height of the cut). The authors show that the exponent τ varies as a function of the height of the cut. This variation is also shown to be dependent on the roughness exponent. These results suggest that the derived relation in [18] between the contour-length distribution exponent and the roughness exponent seems is valid only at a certain height. An investigation of this dependence is planned for future work.

In the rest of the paper we investigate the conformal symmetry of the coastlines of cuts made at different heights of two-dimensional saturated KPZ surface. Theory of SLE provides an appropriate approach to check conformal properties of the geometrical features of such systems. Looking at the fractal dimension obtained for the KPZ coastlines at different level sets, it agrees with the SLE curves of fractal dimension $D_f=1+\kappa/8$, with $\kappa=8/3$, conjectured to describe the scaling limit of SAWs. In [5], we checked various consistencies between the coastlines and both SAWs and $\mathrm{SLE}_{8/3}$. Such coastlines can statistically be defined as the outer boundary of the random walk and of percolation clusters. In our case, since the dimension of the islands is 2, it suggests that they are compact unlike the clusters in critical percolation.

Statistical behavior of the coastlines here is similar to the statistics of rocky shorelines studied recently in [22]. The winding-angle statistics of the shorelines is consistent with the prediction of $\mathrm{SLE}_{8/3}$. To be more serious about the similarities between the 2D-KPZ coastlines and rocky shorelines and, moreover, giving another justification for conformal invariancy, we compute the winding-angle statistics for the contour lines of 2D-KPZ surface.

Duplantier and Saleur studied in [23] the winding angle between the two end points of a finite SAW in two dimensions. Using Coulomb gas methods, they found that the distribution of winding angle is Gaussian with the winding variance of $\sim(8/g)\ln L$, where L is the distance between the end points and g is Coulomb gas coupling parameter which is related to κ by $g=4/\kappa$. They have also shown that the winding angle at a single end point relative to the global average direction of the curve is a Gaussian with variance of $(4/g)\ln L$. Wieland and Wilson found in [24] that the variance in the winding at typical points along the curve is 1/4 as large as the variance in the winding at the end points.

We define the winding angle θ (as used in [22]), as the angle between the line joining two points separated by a length l along the curve and the local tangent in the reference point, measured counterclockwise in *radian*. For conformally invariant curves of diffusivity κ, its variance behaves like [22]

$$\langle\theta^2\rangle\sim\frac{2\kappa}{8+\kappa}\ln l, \tag{3}$$

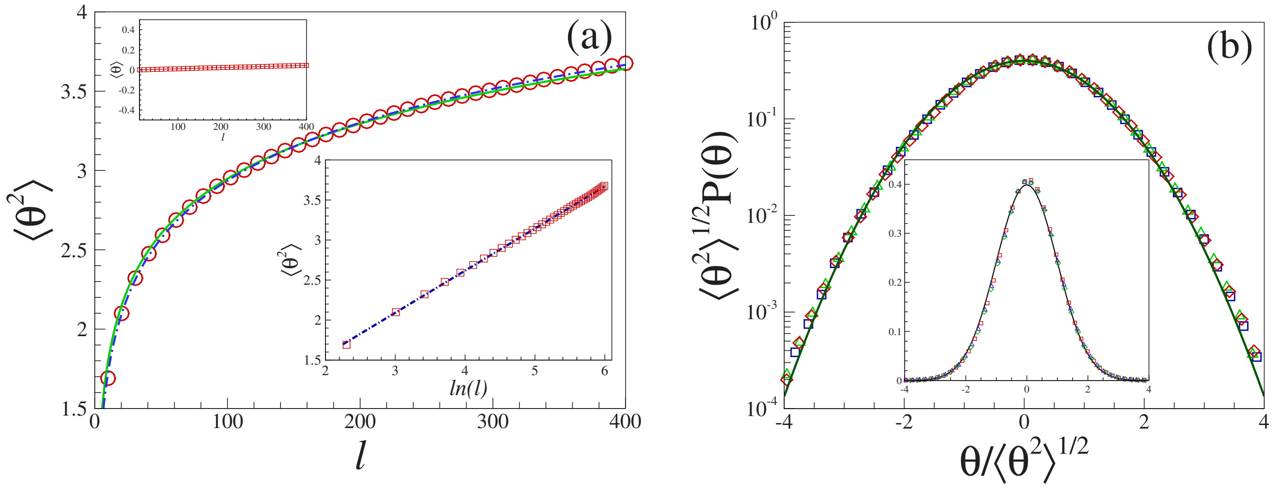

FIG. 5. (Color online) Winding-angle statistics of isoheight lines of 2D-KPZ surface simulated on square lattice of size 2048^2. (a) Main: logarithmic behavior of the variance of the winding angle as a function of distance l along the curve [see Eq. (3)]. Upper-left inset: the mean winding angle as a function of l. Lower-right inset: semilogarithmic behavior of the winding. Dotted-dashed lines show the best fit according to Eq. (3), with $\kappa=2.76\pm0.1$. The solid line in the main frame shows the fit with $\kappa=8/3$ for comparison with SAWs. (b) The rescaled probability density function of the winding angle at lengths $l=50$, 200, 400 in units of lattice spacing compared to the standard Gaussian density,

where the average is taken over an ensemble of 1800 curves by moving the reference point along each curve.

As shown in Fig. 5(a), the variance of winding angle for isoheight lines at a cut made at mean height has a logarithmic behavior. The best fit (dotted dashed lines), corresponds to $\kappa=2.76\pm0.1$ which is compared with the fit by setting $\kappa=8/3$ for SAWs (solid line). The length scale, l, is measured in units of lattice spacing which is set to unity, on square sample size of 2048^2. For each configuration the largest loop is selected, so within the length scale l, the curves do not have a preferred direction and the mean winding angle is zero [upper-left inset of Fig. 5(a)]. The rescaled probability density function of the winding angle at different lengths is shown in Fig. 5(b), which converges to a standard Gaussian density. The results are consistent with [5]. We find no changes in the winding statistics of contour lines at different level sets.

In conclusion, studying the statistics of isoheight lines of saturated 2D-KPZ surface at different level sets we find that the fractal dimensions of cluster heights and their perimeter remain unchanged when changing the height of the cut. We also find that the exponent associated with the distribution function of the cluster size (the mass of clusters is considered here) changes as a function of the height of the cut. It linearly decreases for cuts made below the mean height and crosses over to an almost linear fluctuation around a specific value above the mean height. We also tested another exponents related to the distribution of length of the loops, the area of the loops, and the radius of gyration which all change at different level cuts (the results are not included in this paper).

The winding-angle statistics of the contour lines also suggests that their statistics is comparable to that of SAWs. This confirms that the conformal invariant property of the contour lines is given by $SLE_{8/3}$.

[1] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London, Ser. A **381**, 17 (1982).
[2] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[3] A. L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, 1995).
[4] J. G. Amar and F. Family, Phys. Rev. A **41**, 3399 (1990); E. Marinari *et al.*, J. Phys. A **33**, 8181 (2000).
[5] A. A. Saberi, M. D. Niry, S. M. Fazeli, M. R. RahimiTabar, and S. Rouhani, Phys. Rev. E **77**, 051607 (2008).
[6] O. Schramm, Isr. J. Math. **118**, 221 (2000).
[7] J. Cardy, Ann. Phys. **318**, 81 (2005); M. Bauer and D. Bernard, Phys. Rep. **432**, 115 (2006); I. A. Gruzberg, J. Phys. A **39**, 12601 (2006); H. C. Fogedby, e-print arXiv:0706.1177.
[8] O. Schramm and S. Sheffield, e-print arXiv:math.PR/0605337.
[9] G. F. Lawler *et al.*, J. Am. Math. Soc. **16**, 917 (2003).
[10] T. Kennedy, Phys. Rev. Lett. **88**, 130601 (2002).
[11] G. F. Lawler, O. Schramm, and W. Werner, in *Proceedings of Symposia in Pure Mathematics*, (American Mathematical Society, Providence, RI, 2004), Vol. 72, Part 2, p. 339.
[12] D. Bernard *et al.*, Nat. Phys. **2**, 124 (2006).
[13] D. Bernard, G. Boffetta, A. Celani, and G. Falkovich, Phys. Rev. Lett. **98**, 024501 (2007).
[14] C. Amoruso, A. K. Hartmann, M. B. Hastings, and M. A.

Moore, Phys. Rev. Lett. **97**, 267202 (2006); D. Bernard, P. LeDoussal, and A. A. Middleton, Phys. Rev. B **76**, 020403(R) (2007).
[15] J. P. Keating, J. Marklof, and I. G. Williams, Phys. Rev. Lett. **97**, 034101 (2006); E. Bogomolny, R. Dubertrand, and C. Schmit, J. Phys. A **40**, 381 (2007).
[16] A. A. Saberi, M. A. Rajabpour, and S. Rouhani, Phys. Rev. Lett. **100**, 044504 (2008).
[17] A. A. Saberi, S. Moghimi-Araghi, H. Dashti-Naserabadi, and S. Rouhani, eprint arXiv:0812.0939.
[18] J. Kondev and C. L. Henley, Phys. Rev. Lett. **74**, 4580 (1995).
[19] J. Kondev, C. L. Henley, and D. G. Salinas, Phys. Rev. E **61**, 104 (2000).
[20] F. D. A. Aarâo Reis, Phys. Rev. E **69**, 021610 (2004).
[21] Z. Olami and R. Zeitak, Phys. Rev. Lett. **76**, 247 (1996).
[22] G. Boffetta *et al.*, Geophys. Res. Lett. **35**, L03615 (2008).
[23] B. Duplantier and H. Saleur, Phys. Rev. Lett. **60**, 2343 (1988).
[24] B. Wieland and D. B. Wilson, Phys. Rev. E **68**, 056101 (2003).